International University

School of Electrical Engineering

PRINCIPLES OF ELECTRICAL ENGINEERING 2

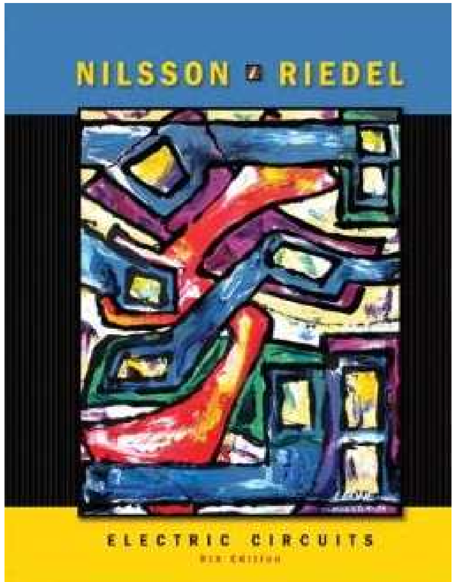
Lecture # 3 & 4: Introduction to the Laplace Transform

Text book:

Electric Circuits

James W. Nilsson & Susan A. Riedel

8th or 9th Edition.



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Objectives

- Be able to calculate the Laplace transform of a function.
- Be able to calculate the inverse Laplace transform.
- Understanding and know how to use the initial value theorem and the final value theorem.

Outline

- Definition of the Laplace transform
- The step function
- The impulse function
- Functional transforms
- Operational transforms
- Inverse transforms
- Poles and Zeros of F(s)
- Initial- and final-value theorems

Pierre-Simon Laplace
(1749–1827)

Laplace transform

- The Laplace transform provides a useful method of solving certain types of differential equations when certain initial conditions are given, especially when the initial values are zero.
- This is powerful analytical technique used to study the behavior of linear, lumped-parameter circuits.
- It is often easier to analyze the circuit in its <u>Laplace form</u>, than to <u>form differential equations</u>.

Definition of the Laplace transform

The Laplace transform is a tool for converting **time-domain** equations into **frequency-domain** equations, for $t > 0$ is defined by the following integral defined over 0 to ∞

L{f(t)} is read the Laplace transform of f(t)

$$\boxed{L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt}$$

The resulting expression is a function of s, which we write as F(s). In words we say "The Laplace Transform of f(t) equals function F of s" and write: L{f(t)} = F(s)

Similarly, the Laplace transform of a function *g*(*t*) would be written:
L{g(t)} = G(s)

Laplace transform

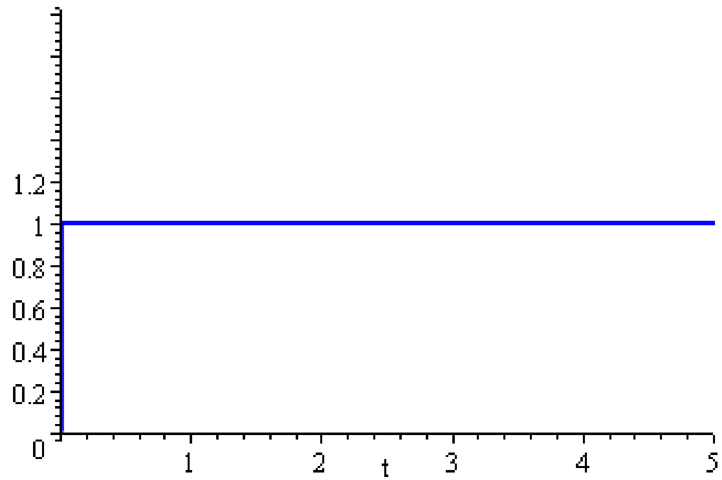
- Create a new domain to make mathematical manipulations easier. After finding the unknown in the new domain, we inverse-transform it back to the original domain.
- In circuit analysis, Laplace transform is used to transform a set of differential equations from the time domain to a set of algebraic equations in the frequency domain ➔ simplify the solution.
- Some sources may not have Laplace transform.
- For *F(s)* is determined by the behavior of *f(t)* only for positive values of *t*, which is referred to as the one-sided, or unilateral Laplace transform. **F(s) is understood to be the one-side transform!**

Laplace transform

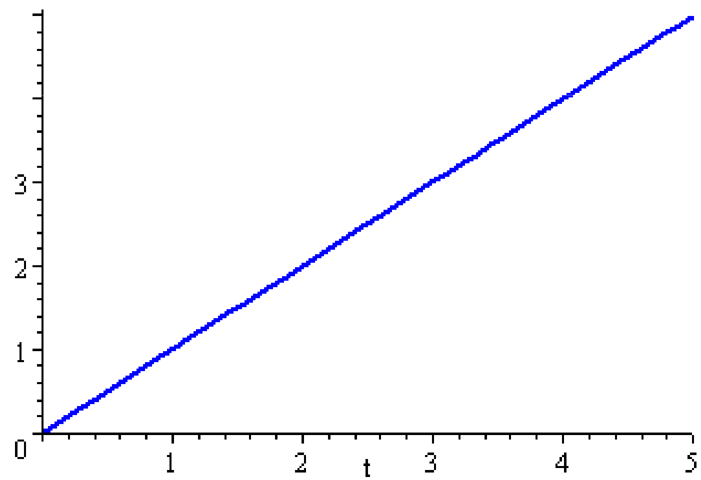
- There are **two types** of Laplace transform:
  - ***Functional transform*** is the Laplace transform of a specific function such as $\sin(\omega t)$; t; e^{-at},…
  - ***Operational transform*** defines a general mathematical property of the Laplace transform

Laplace transform

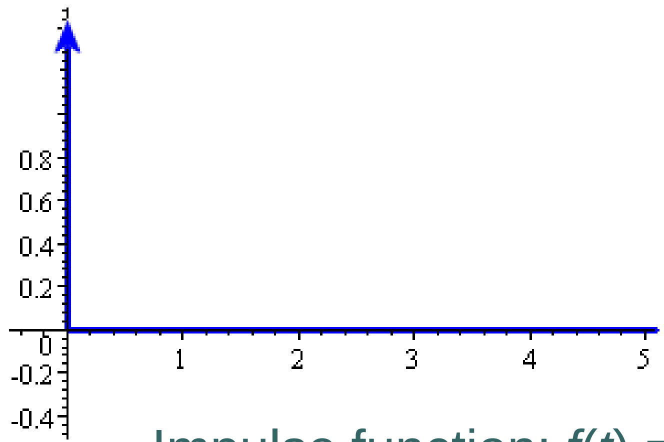
Reminder: Unit, Ramp and Impulse Functions



Unit step function: $f(t) = u(t)$



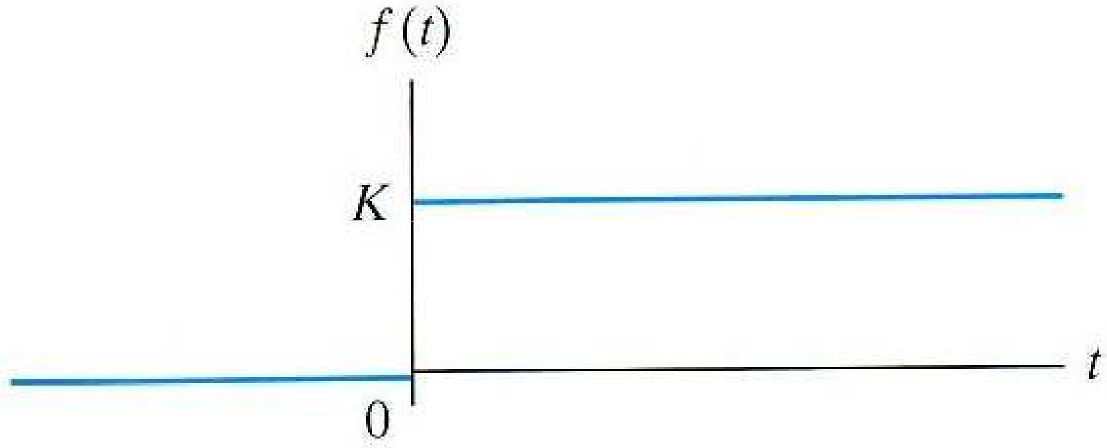
Ramp function: $f(t) = (t)$



Impulse function: $f(t) = \delta(t)$

$\delta(t)$ represents an impulse at $t = 0$ and has value 0 otherwise.

The Step Function: Ku(t)



The step function *Ku(t)* describes a function that experiences a discontinuity from one constant level to another at some point in time.

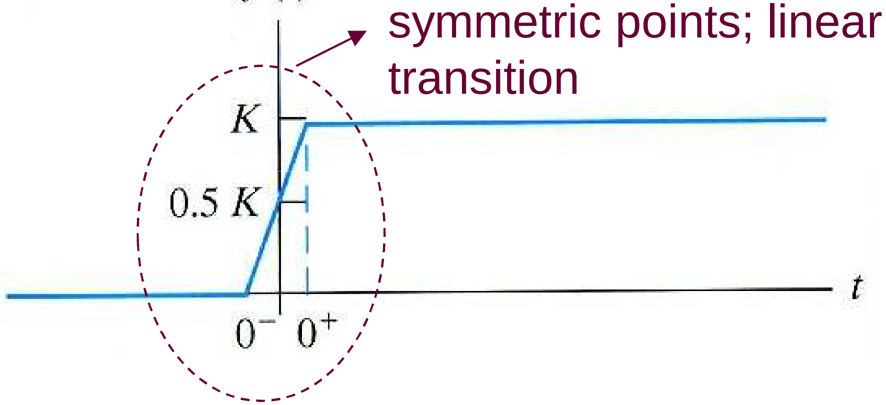
$$Ku(t) = 0 \qquad , t < 0$$

$$Ku(t) = K \qquad , t > 0$$

K is the magnitude of the jump.
If $K = 1$, *Ku(t)* is the unit step function.

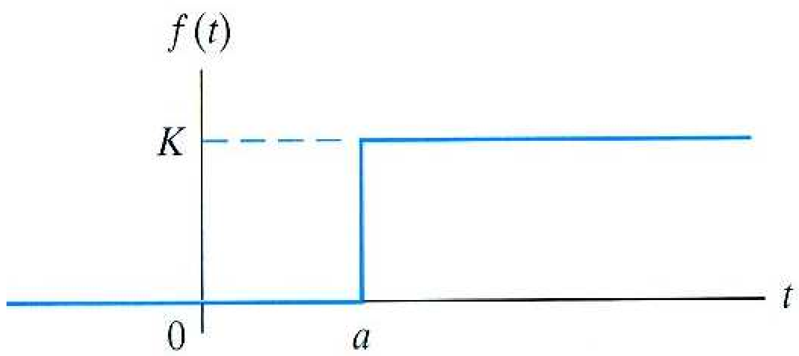
The Step Function: Ku(t)

0^- and 0^+ represent symmetric points; linear transition



When the step function is not defined at t = 0, it is necessary to define the transition between 0^- and 0^+ ,

ex.: ***Ku(0) = 0.5K***



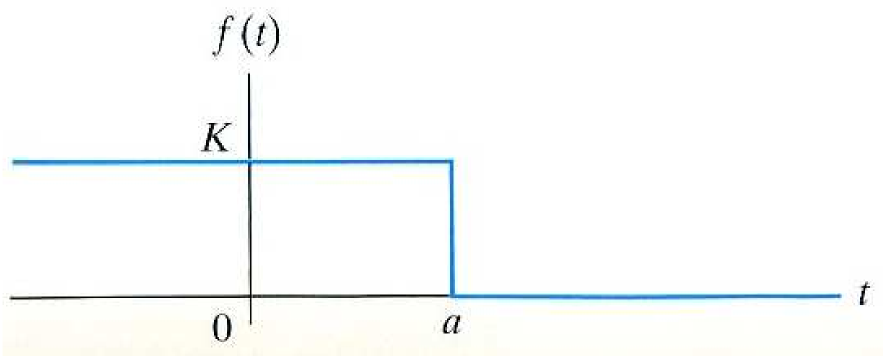
When discontinuity occur @ t ≠ 0, it is expressed:

Ku(t – a) = 0 , t < a

Ku(t – a) = K , t > a

Note: step function = 0 when t-a < 0

step function = K when t-a > 0



Ku(a – t) = K , t < a

Ku(a – t) = 0 , t > a

The Step Function: Ku(t)

- Example: Using Step Functions to Represent a Function of Finite Duration

Use step functions to write an expression for the function illustrated in Fig. 12.6.

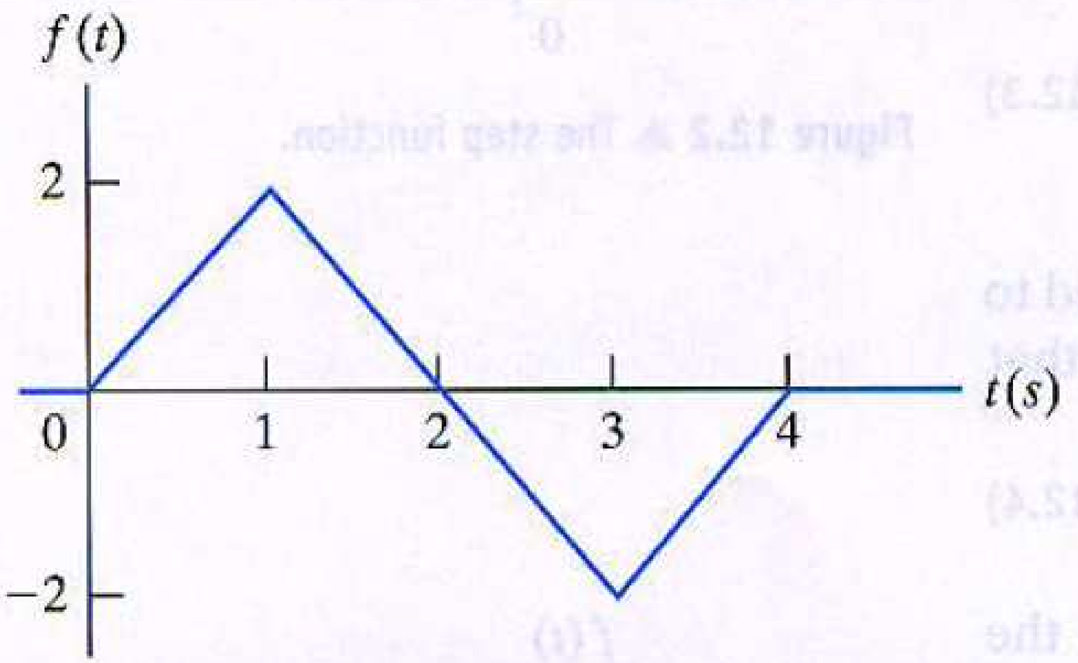


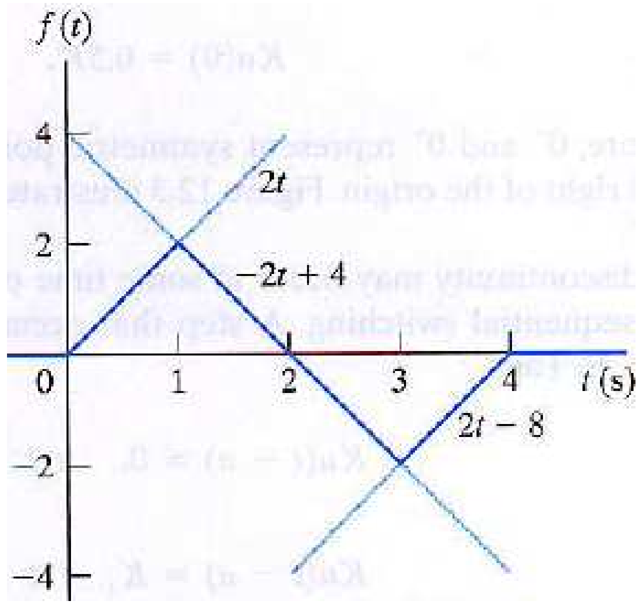
Figure 12.6 ▲ The function for Example 12.1.

The Step Function: Ku(t)

- Solution

Linear segments have break points at 0, 1, 3, and 4 s

Use the step function to initiate & terminate these linear segments @ the proper times



Equations: (+2t) on @ t = 0, off @ = 1

(-2t + 4) on @ t = 1, off @ t = 3

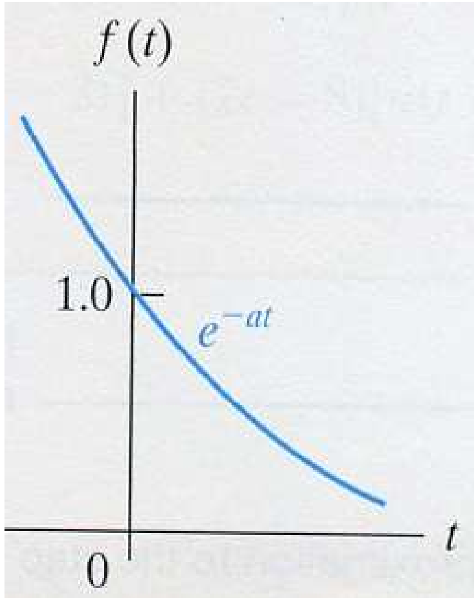
(+2t – 8) on @ t = 3 , off @ t = 4

The expression for f(t) is

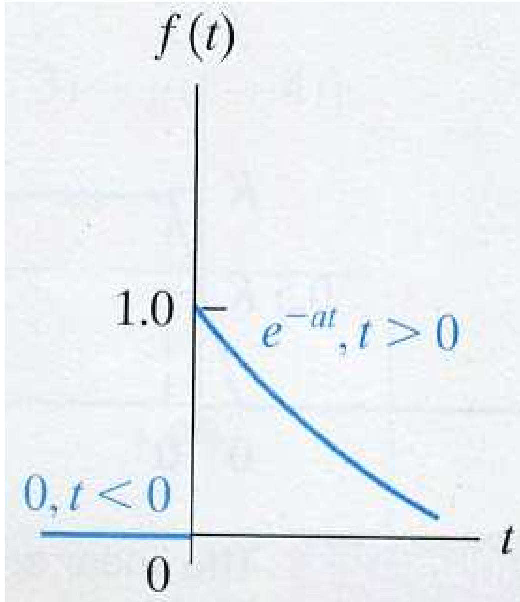
$f(t) = 2t[u(t) - u(t-1)] + (-2t + 4)[u(t-1) - u(t-3)] + (2t - 8)[u(t-3) - u(t-4)].$

The Impulse Function

- **Continuous and discontinuous function**



f(t) is continuous @ the origin



f(t) is discontinuous @ the origin

- The concept of *an impulse function* enables us to define the derivative at a discontinuity, and thus to define the Laplace transform of that derivative.

The Impulse Function

- **The impulse function Kδ(t) is defined:**

$$\int_{-\infty}^{\infty} K\delta(t)\,dt = K$$

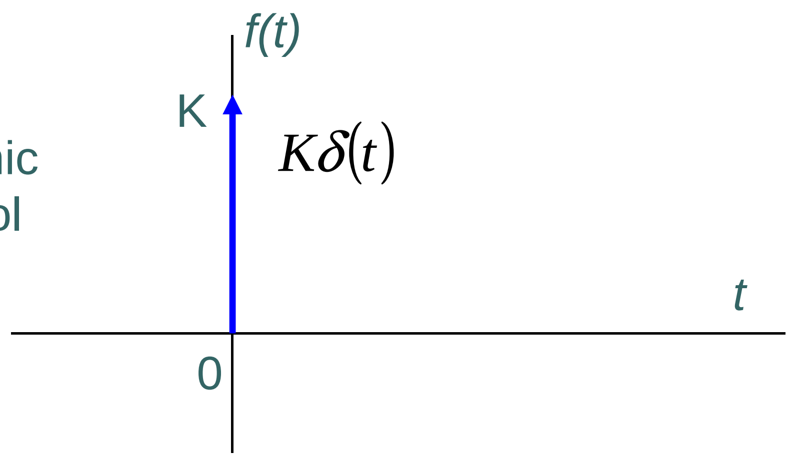
$$\delta(t) = 0, \qquad t \neq 0$$

K is the strength of the impulse.
If ***K* = 1**, *Kδ(t)* is the **unit impulse function**.

Must has 3 characteristics:

1. *An impulse is a signal of infinite amplitude and zero duration*
2. *The area under the impulse function is constant.*
3. *The impulse is zero everywhere except @ t = 0*

Graphic symbol

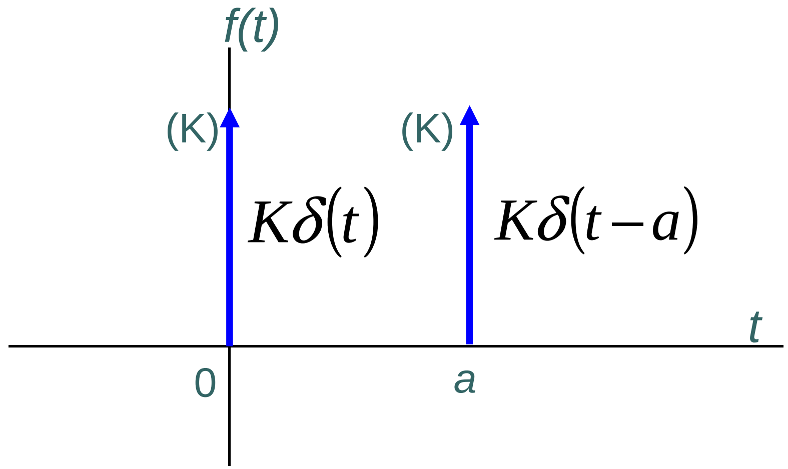


Impulsive voltages and currents occur in circuit analysis either because of a switching operation or because the circuit is excited by an impulsive source.

The Impulse Function

- **Sifting property:**

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$



The impulse function sifts out everything except the value of f(t) at t = a

- Use the sifting property of the impulse function to find its Laplace transform:

$$L\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

Functional Transforms

- A functional transform is the Laplace transform of a specific function of t.

  Example:

  Laplace transform of the unit step function:

$$L\{u(t)\} = \int_{0^+}^{\infty} f(t)e^{-st}dt = \frac{1}{s}$$

  Laplace transform of the decaying exponential function:

$$L\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}dt = \frac{1}{s+a}$$

Important functional transform pairs

Type	f(t) (t > 0⁻)	F(s)
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Operational Transforms

- Operational transforms define the general mathematical properties of the Laplace transform.
- The operations of primary interest include:

(1) Multiplication by a constant

*Multiplication of **f(t)** by a constant corresponds to multiplying **F(s)** by the same constant.*

$$L\{Kf(t)\} = KF(s)$$

(2) Addition (subtraction)

Addition (subtraction) in the time domain translates in to addition (subtraction) in the frequency domain.

$$L\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s)$$

Operational Transforms

(3) Differentiation

*Differentiation in the time domain corresponds to multiplying **F(s)** by **s** and then subtracting the initial value of **f(t)**.*

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

*It can be seen that differentiation in the time domain reduces to an algebraic operation in the **s** domain.*

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}\,dt.$$ Letting u = e^{-st} & dv = [df(t)/dt]dt

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = e^{-st}f(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)(-se^{-st}dt).$$ The evaluation of $e^{-st}f(t)$ @ t = ∞ is 0

$$\Rightarrow \quad -f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st}dt = sF(s) - f(0^-)$$

Operational Transforms

(4) Integration

Integration in the time domain corresponds to dividing by s in the s domain.

$$L\left\{\int_{0^-}^{t} f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

Operation of integration in the time domain is transformed to the algebraic operation of multiplying by 1/s in the s domain.

Laplace transform translates a set of differential equations into a set of algebraic equations.

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)\,dx\right]e^{-st}\,dt$$ letting $$\begin{cases} u = \int_{0^-}^{t} f(x)\,dx, \\ dv = e^{-st}\,dt. \end{cases} \Rightarrow \quad du = f(t)\,dt, \quad v = -\frac{e^{-st}}{s}.$$

integration-by-parts formula yields:

$$\mathcal{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \underbrace{-\frac{e^{-st}}{s}\int_{0^-}^{t} f(x)\,dx\bigg|_{0^-}^{\infty}}_{0} + \int_{0^-}^{\infty}\frac{e^{-st}}{s}f(t)\,dt = \frac{F(s)}{s}$$

Operational Transforms

(5) Translation in the time domain (time shifting)

Translation in the time domain corresponds to multiplication by an exponential in the frequency domain.

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a > 0$$

(6) Translation in the frequency domain (frequency shifting)

Translation in the frequency domain corresponds to multiplication by an exponential in the time domain.

$$L\{e^{-at}f(t)\} = F(s+a)$$

(7) Scale changing

The scale-change property gives the relationship between f(t) and F(s) when the time variable is multiplied by a positive constant.

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$$

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^{\infty} F(u)\,du$

List of Operational Transforms

Inverse Transforms

- In linear lumped-parameter circuits, F(s) is a rational function of s.
- Rational function can be expressed in the form of a ratio of two polynomials

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

  - *a and b are real constants*
  - *m and n are positive integers*

- F(s) is a *proper rational function* **if** m > n. The inverse transform is found by a partial fraction expression.
- F(s) is an *improper rational function* **if** m ≤ n. It can be inverse-transformed by first expanding it into a sum of a polynomial and a proper rational function.

Inverse Transforms
Proper Rational Functions

- A proper rational function is expanded into a sum of partial fractions by writing a term or a series of terms for each root of D(s)

For example: $\frac{s+6}{s(s+3)(s+1)^2}$ ⇨ Denominator has four roots!

$$\frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

- Inverse transforms are found lies in recognizing the f(t) corresponding to each term in the sum of partial fractions.

⇨ $$L\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\} = (K_1 + K_2 e^{-3t} + K_3 t e^{-t} + K_4 e^{-t})u(t)$$

Inverse Transforms

Distinct Real Roots of *D(s)*

- To find a K associated with a term that arises because of a distinct root of D(s):
  - *multiply both sides of the identity by a factor equal to the denominator beneath the desired K.*
  - *Evaluate both sides of the identity at the root corresponding to the multiplying factor, the right-hand side is always the desired K, and the left-hand side is always its numerical value.*

Example:

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

Inverse Transforms

Distinct Real Roots of *D(s)*

Example:

To find the value of K_1, we multiply both sides by s and then evaluate both sides at $s = 0$:

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2 s}{s+8}\right|_{s=0} + \left.\frac{K_3 s}{s+6}\right|_{s=0},$$

or

$$\frac{96(5)(12)}{8(6)} \equiv K_1 = 120.$$

To find the value of K_2, we multiply both sides by $s + 8$ and then evaluate both sides at $s = -8$:

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8}$$

$$\equiv \left.\frac{K_1(s+8)}{s}\right|_{s=-8} + K_2 + \left.\frac{K_3(s+8)}{(s+6)}\right|_{s=-8},$$

or

$$\frac{96(-3)(4)}{(-8)(-2)} = K_2 = -72.$$

Inverse Transforms
Distinct Real Roots of *D(s)*

Example:

Then K_3 is

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = K_3 = 48.$$

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}$$

Now confident that the numerical values of the various Ks are correct, we proceed to find the inverse transform:

$$\mathscr{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t)$$

Inverse Transforms

Distinct Complex Roots of *D(s)*

- The procedure is the same as for distinct real roots.
- The only difference is that the algebra in the distinct complex roots involves complex numbers.
- Note:
  - In physical realizable circuits, complex roots always appear in conjugate pairs.
  - The coefficients associated with these conjugate pairs are themselves conjugates.
  - Therefore, we just need to calculate only half the coefficients.

Example:

$$F(s)=\frac{100(s+3)}{(s+6)(s^2+6s+25)}=\frac{K_1}{s+6}+\frac{K_2}{s+3-j4}+\frac{K_3}{s+3+j4}$$

Inverse Transforms
Distinct Complex Roots of *D(s)*

Example:

find the roots of the quadratic term $s^2 + 6s + 25$:

$$s^2 + 6s + 25 = (s + 3 - j4)(s + 3 + j4).$$

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

To find K_1, K_2, and K_3, we use the same process as before:

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12,$$

a + jb = r(cosβ + sinβ)

$$r = (a^2 + b^2)^{1/2}$$

$e^{j\beta} = \cos\beta + j\sin\beta$

$e^{-j\beta} = \cos\beta - j\sin\beta$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^\circ},$$

Inverse Transforms

Distinct Complex Roots of *D(s)*

Example:

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^\circ}.$$

Finally:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\angle -53.13^\circ}{s+3-j4} + \frac{10\angle 53.13^\circ}{s+3+j4}.$$

We now proceed to inverse-transform

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t}$$

$$+ 10e^{j53.13^\circ}e^{-(3+j4)t})u(t)$$

$$= [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t)$$

Inverse Transforms

Repeated Real Roots of *D(s)*

- To find the coefficients associated with the terms generated by a multiple root of multiplicity *r*, multiply both sides of the identity by the multiple root raised to its *r* th power.
- *K* appearing over the factor raised to the *r* th power is found by evaluating both sides of the identity at the multiple root.
- The remaining *(r – 1)* coefficients are found by differentiating both sides of the identity *(r – 1)* times.
- At the end of each differentiation, evaluate both sides of the identity at the multiple root.
- The right-hand side is always the desired *K*, and the left-hand side is always its numerical value.

Example: $$F(s) = \frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5} \qquad (*)$$

We find K_1 as previously described; that is,

$$K_1 = \left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0} = \frac{100(25)}{125} = 20$$

To find K_2, we multiply both sides by $(s+5)^3$ and then evaluate both sides at -5:

$$\left.\frac{100(s+25)}{s}\right|_{s=-5} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + K_3(s+5)\Big|_{s=-5} + K_4(s+5)^2\Big|_{s=-5}$$

$$\frac{100(20)}{(-5)} = K_1 \times 0 + K_2 + K_3 \times 0 + K_4 \times 0 = K_2 = -400$$

To find K_3 we first must multiply both sides of Eq.(*) by $(s+5)^3$. Next we differentiate both sides once with respect to s and then evaluate at $s = -5$:

$$\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5} = \frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5} + \frac{d}{ds}[K_2]_{s=-5} + \frac{d}{ds}[K_3(s+5)]_{s=-5}$$

$$+ \frac{d}{ds}[K_4(s+5)^2]_{s=-5},$$

$$100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = K_3 = -100.$$

To find K_4 we first multiply both sides of Eq. **(*)** by $(s+5)^3$. Next we differentiate both sides twice with respect to s and then evaluate both sides at $s = -5$. After simplifying the first derivative, the second derivative becomes

$$100\frac{d}{ds}\left[-\frac{25}{s^2}\right]_{s=-5} = K_1\frac{d}{ds}\left[\frac{(s+5)^2(2s-5)}{s^2}\right]_{s=-5} + 0 + \frac{d}{ds}[K_3]_{s=-5} + \frac{d}{ds}[2K_4(s+5)]_{s=-5},$$

$$-40 = 2K_4. \qquad K_4 = -20.$$

$$\frac{100(s+25)}{s(s+5)^3} = \frac{20}{s} - \frac{400}{(s+5)^3} - \frac{100}{(s+5)^2} - \frac{20}{s+5}.$$

The inverse transform

$$\mathscr{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\} = [20 - 200t^2e^{-5t} - 100te^{-5t} - 20e^{-5t}]u(t).$$

Inverse Transforms

Repeated Complex Roots of *D(s)*

- The procedure is the same as for the repeated real roots.
- The only difference is that the algebra involves complex numbers
- Because complex roots always appear in conjugate pairs and the coefficients associated with a conjugate pair are also conjugates, only half the *K* s need to be evaluated

Example:

$$F(s) = \frac{768}{(s^2+6s+25)^2} = \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$= \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{(s+3-j4)} + \frac{K_1^*}{(s+3+j4)^2} + \frac{K_2^*}{(s+3+j4)}$$

Now we need to evaluate only K_1 and K_2, because K_1^* and K_2^* are conjugate values. The value of K_1 is

$$K_1 = \left.\frac{768}{(s + 3 + j4)^2}\right|_{s=-3+j4} = \frac{768}{(j8)^2} = -12$$

The value of K_2 is

$$K_2 = \frac{d}{ds}\left[\frac{768}{(s + 3 + j4)^2}\right]_{s=-3+j4} = -\left.\frac{2(768)}{(s + 3 + j4)^3}\right|_{s=-3+j4} = -\frac{2(768)}{(j8)^3}$$

$$= -j3 = 3\angle -90^\circ.$$

$$K_1^* = -12, \quad K_2^* = j3 = 3\angle 90^\circ$$

We now group the partial fraction expansion by conjugate terms to obtain

$$F(s) = \left[\frac{-12}{(s + 3 - j4)^2} + \frac{-12}{(s + 3 + j4)^2}\right] + \left(\frac{3\angle -90^\circ}{s + 3 - j4} + \frac{3\angle 90^\circ}{s + 3 + j4}\right).$$

We now write the inverse transform of $F(s)$: $f(t) = [-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t - 90^\circ)]u(t).$

Note that if $F(s)$ has a real root a of multiplicity r in its denominator, the term in a partial fraction expansion is of the form

$$\frac{K}{(s + a)^r}.$$

The inverse transform of this term is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+a)^r}\right\} = \frac{Kt^{r-1}e^{-at}}{(r-1)!}u(t).$$

If $F(s)$ has a complex root of $\alpha + j\beta$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}.$$

The inverse transform of this pair is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}\right\} = \left[\frac{2|K|t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t)$$

Inverse Transforms

Four Useful Transform Pairs

Pair number	Nature of roots	F(s)	f(t)
1	Distinct real	$\dfrac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\dfrac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\dfrac{K}{s+\alpha-j\beta}+\dfrac{K^*}{s+\alpha+j\beta}$	$2\|K\|e^{-\alpha t}cos(\beta t+\theta)u(t)$
4	Repeated complex	$\dfrac{K}{(s+\alpha-j\beta)^2}+\dfrac{K^*}{(s+\alpha+j\beta)^2}$	$2t\|K\|e^{-\alpha t}cos(\beta t+\theta)u(t)$

Inverse Transforms

Improper Rational Functions

- An improper rational function can always be expanded into a polynomial plus a proper rational function.
- The polynomial is inverse-transformed into impulse functions and derivatives of impulse functions.
- The proper rational function is inverse-transformed by the techniques outlined in previous section

Example:

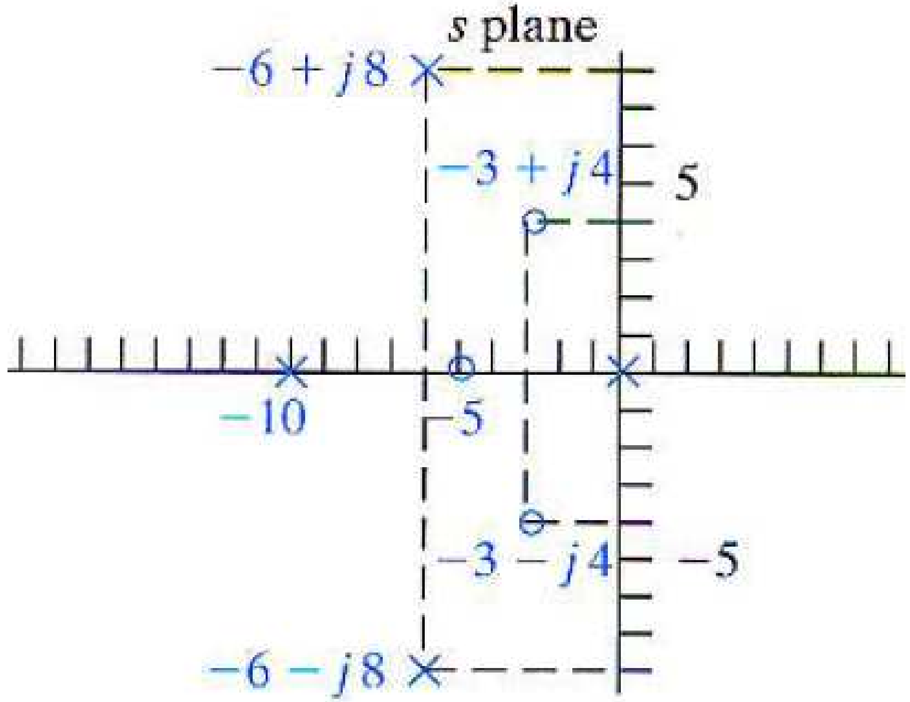
$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 - \frac{20}{s+4} + \frac{50}{s+5}$$

Poles and Zeros of *F(s)*

- F(s) can be expressed as the ratio of two factored polynomials.
- The roots of the denominator are called poles and are plotted as Xs on the complex s plane.
- The roots of the numerator are called zeros and are plotted as 0s on the complex s plane.
- (see page 494 & 495)



Plotting poles and zeros on the s plane.

Initial and Final Value Theorems

The initial- and final- value theorems are used because they enable us to determine from F(s) the behavior of f(t) at 0 and ∞

- The initial-value theorem states that:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) \qquad (*)$$

The theorem assumes that f(t) contains no impulse functions.

- The final-value theorem states that:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \qquad (**)$$

The theorem is valid only if the poles of F(s), except for a first- order pole at the origin, lie in the left half of the s plane.

Initial and Final Value Theorems

Prove (*)

$$\mathscr{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\,dt.$$

Now we take the limit as $s \to \infty$:

$$\lim_{s\to\infty} [sF(s) - f(0^-)] = \lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\,dt.$$

Observe that the right-hand side of Eq. 12.96 may be written as

$$\lim_{s\to\infty}\left(\int_{0^-}^{0^+} \frac{df}{dt} e^{0} dt + \int_{0^+}^{\infty} \frac{df}{dt} e^{-st}\,dt\right).$$

As $s \to \infty$, $(df/dt)e^{-st} \to 0$ The first integral reduces to $f(0^+) - f(0^-)$, which is independent of s.

$$\lim_{s\to\infty} \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\,dt = f(0^+) - f(0^-).$$

Initial and Final Value Theorems

Prove (*)

Because $f(0^-)$ is independent of s, the left-hand side of Eq.

$$\lim_{s\to\infty}[sF(s) - f(0^-)] = \lim_{s\to\infty}[sF(s)] - f(0^-).$$

$$\lim_{s\to\infty} sF(s) = f(0^+) = \lim_{t\to 0^+} f(t),$$

which completes the proof of the initial-value theorem.

Prove (**) we take the limit as $s \to 0$: $\lim_{s\to 0}[sF(s) - f(0^-)] = \lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right)$

$$\lim_{s\to 0}\left(\int_{0^-}^{\infty}\frac{df}{dt}e^{-st}\,dt\right) = \int_{0^-}^{\infty}\frac{df}{dt}\,dt.$$

Because the upper limit on the integral is infinite, this integral may also be written as a limit process:

Initial and Final Value Theorems

Prove (**)

$$\int_{0^-}^{\infty} \frac{df}{dt} dt = \lim_{t \to \infty} \int_{0^-}^{t} \frac{df}{dy} dy,$$

where we use y as the symbol of integration to avoid confusion with the upper limit on the integral. Carrying out the integration process yields

$$\lim_{t \to \infty} [f(t) - f(0^-)] = \lim_{t \to \infty} [f(t)] - f(0^-).$$

$$\Rightarrow \quad \lim_{s \to 0} [sF(s)] - f(0^-) = \lim_{t \to \infty} [f(t)] - f(0^-).$$

$$\Rightarrow \quad \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

Initial and Final Value Theorems - Application

Consider the transform pair given by Eq.

$$\mathcal{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t).$$

The initial-value $$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{100s^2[1+(3/s)]}{s^3[1+(6/s)][1+(6/s)+(25/s^2)]} = 0,$$

$$\lim_{t\to 0^+} f(t) = [-12 + 20\cos(-53.13^\circ)](1) = -12 + 12 = 0.$$

The final-value theorem gives $$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{100s(s+3)}{(s+6)(s^2+6s+25)} = 0,$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t) = 0.$$